

Quantization of Higher Dimensional Linear Dilaton Black Hole Area/Entropy From Quasinormal Modes

I. Sakalli*

*Department of Physics, Eastern Mediterranean University,
G. Magusa, North Cyprus, via Mersin 10, Turkey.*

The quantum spectra of area and entropy of higher dimensional linear dilaton black holes in various theories via the quasinormal modes method are studied. It is shown that quasinormal modes of these black holes can reveal themselves when a specific condition holds. Finally, we obtain that a higher dimensional linear dilaton black hole has equidistant area and entropy spectra, and both of them are independent on the spacetime dimension.

I. INTRODUCTION

Inspired by Giddings and Strominger [1], Clément et al. [2] proposed a new four-dimensional static spherically symmetric solution describing a black hole in a linear dilaton background so-called linear dilaton black hole (LDBH). The LDBHs of [2] are exact solutions to Einstein-Maxwell-Dilaton (EMD) theory. In general, LDBHs represent non-asymptotically flat (NAF) spacetimes and break all supersymmetries. Since the LDBHs are NAF spacetimes, their physical mass can be computed by the Brown-York formalism [3], and shown to satisfy the first law of thermodynamics consistent with the definition of the Hawking temperature and of the geometric entropy, which is the quarter of the horizon area [4, 5]. The most remarkable features of the LDBHs are that the temperature is constant (representing an isothermal process) and the mass is independent of the temperature so that the heat capacity becomes zero.

A long time ago, higher dimensional generalization of the LDBHs was inapparently given in EMD theory by Chan et al. [6] as some limit of extremal dilaton black holes [7]. Recently, it has been shown that higher dimensional static LDBH solutions exist in Einstein-Yang-Mills-Dilaton (EYMD) [8] and Einstein-Yang-Mills-Born-Infeld-Dilaton (EYMBID) [9] theories. A study of the thermodynamic properties of these black holes has also been done [10]. Surprisingly, however, there are no detailed studies of the entropy/area spectra and the quasinormal modes (QNMs – the characteristic ringing frequencies) of the higher dimensional LDBHs. The motivation of this paper is to fill such a gap.

The quantum properties of black holes have attracted much attention for many years. Firstly, Bekenstein [11–15] presumed that the black hole horizon area and entropy ought to be quantized. Bekenstein proposed that the black hole horizon area is an adiabatic invariant, and has a discrete and equally spaced spectrum

$$\mathcal{A}_n = \epsilon \hbar n \quad (n = 0, 1, 2, \dots), \quad (1)$$

where \mathcal{A}_n is the area of the black hole horizon and n is the quantum number. ϵ can be considered as a numerical coefficient of the order of unity when all the fundamental constants except Planck's constant (\hbar) have been set to one. In this regard, Hod [16] made a semiclassical study, based on Bohr's correspondence principle (the reader may refer to [17]), that the coefficient ϵ can be determined by manipulating the QNM frequencies of a vibrating black hole. The Hod's result of ϵ was also obtained by Kunstatter [18] who used the method of adiabatic invariant together with the Bohr-Sommerfeld quantization condition. Essentially, Kunstatter showed that a natural adiabatic invariant for system with energy E and vibrational frequency $\Delta\omega(E)$ is given by

$$I_{adb} = \int \frac{dE}{\Delta\omega(E)}, \quad \Delta\omega = \omega_{n+1} - \omega_n. \quad (2)$$

and it has an equidistant spectrum $I_{adb} \simeq n\hbar$ at large quantum numbers. Maggiore [19] has recently extended the Kunstatter's approach to determine the area spectrum of a black hole. According to the Maggiore's argument, a black hole can be considered as a damped harmonic oscillator. The proper frequency of the equivalent harmonic oscillator corresponds to the QNM frequency (ω), which plays an important role in finding the entropy spectrum. Maggiore

*Electronic address: izzet.sakalli@emu.edu.tr

stated that the QNM frequencies should be of the form $\omega = (\omega_R^2 + \omega_I^2)^{\frac{1}{2}}$, where ω_R and ω_I are the real and imaginary parts of the frequency of the QNM. In the large n limit, $\omega_I \gg \omega_R$. Consequently one has to use ω_I rather than ω_R in the adiabatic quantity. Meanwhile, for a black hole E is identified with the mass (M) of the black hole in expression (2).

Inspired by Maggiore's idea Vagenas and Medved [20, 21] obtained the area spectrum of rotating black holes. However, their results showed that the area spectrum of a rotating black hole is not equidistant. Similar to the studies of Vagenas and Medved, today one can see numerous studies using Maggiore's proposal in the literature (see for instance [22–25]). More recently, Wei et al. [26] have conjectured that for static chargeless black holes, which belong to Einstein's gravity theory the spacing of both entropy and area spectra is equidistant and independent of the dimension of spacetime. In the present paper, we will examine whether Wei et al.'s conjecture is valid for higher dimensional LDBHs or not.

The organization of the paper is as follows. Sect. 2 is devoted to a short review of the higher dimensional LDBHs and their scalar perturbation. In Sect. 3, we apply the semiclassical method to higher dimensional LDBHs and obtain the entropy/area spectrum. Finally we give the concluding remarks.

II. MASSLESS KLEIN GORDON EQUATION IN HIGHER DIMENSIONAL LDBHs

In order to find the quantum of entropy (or area) spectrum by using Maggiore's proposal, first we shall make the calculation of the massless scalar wave equation on N -dimensional LDBHs and obtain the Zerilli type wave equation [27].

Let us first recall the N -dimensional ($N \geq 4$) LDBHs. As it was shown in [10], their metrics are given by

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + R^2 d\Omega_{N-2}^2, \quad (3)$$

with the metric functions

$$f = \Sigma r \left[1 - \left(\frac{r_+}{r} \right)^{\frac{N-2}{2}} \right], \quad R = A\sqrt{r}, \quad (4)$$

It is obvious that metric (3) represents a static, non-rotating BH with a horizon at r_+ . The constants Σ and A in the metric functions (4) take different values according to the concerned theory (EMD, EYMD or EYMBID) [10]. For $r_+ \neq 0$, the horizon hides the naked singularity at $r = 0$. However, in the extreme case of $r_+ = 0$, the central null singularity at $r = 0$ is marginally trapped in which it does not allow outgoing signals to reach external observers. Namely, even in the extreme case of $r_+ = 0$, metric (3) maintains its BH property.

As mentioned before here we consider a massless scalar field satisfying the wave equation $\square\Psi = 0$ in the higher dimensional LDBH spacetime (3) where the metric functions to be used are chosen as in (4). In general, the Laplacian operator on a N -dimensional metric is given by

$$\square = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} \partial^i), \quad i = 1 \dots N, \quad (5)$$

We look for a solution to this wave equation of the form,

$$\Psi = \rho(r) r^{\frac{2-N}{4}} e^{i\omega t} Y_l(\Omega_{N-2}), \quad \text{Re}(\omega) > 0, \quad (6)$$

in which $Y_l(\Omega_{N-2})$ is the eigenfunction of $N - 2$ dimensional Laplace-Beltrami operator ∇_{N-2}^2 with the eigenvalue $-l(l+N-3)$ [28]. After substituting harmonic eigenmodes (6) into the wave equation (5) and making a straightforward calculation, one obtains the following Zerilli equation [27],

$$\left[-\frac{d^2}{dr^{*2}} + V(r) \right] \rho(r) = \omega^2 \rho(r), \quad (7)$$

where the effective potential is given by

$$V(r) = f(r) \left[\frac{l(l+N-3)}{A^2 r} + \frac{(N-2)(N-6)f(r)}{16r^2} + \frac{(N-2)f'(r)}{4r} \right], \quad (8)$$

and the tortoise coordinate is defined as,

$$r^* = \int \frac{dr}{f(r)}, \quad (9)$$

which yields

$$r^* = \frac{2}{\Sigma(N-2)} \ln \left[\left(\frac{r}{r_+} \right)^{\frac{N-2}{2}} - 1 \right]. \quad (10)$$

In principle Zerilli equation (7) can be solved with a particular set of boundary conditions. It can be easily checked that the effective potential (8) vanishes at the horizon ($r^* \rightarrow -\infty$) and becomes $\Sigma \left[\frac{l(l+N-3)}{A^2} + \frac{\Sigma(N-2)^2}{16} \right]$ at spatial infinity ($r^* \rightarrow \infty$). The latter results will be used in the next section in order to obtain the QNMs of the LDBHs.

III. AREA/ENTROPY QUANTIZATION OF HIGHER DIMENSIONAL LDBHS VIA QNM METHOD

In this section, our main goal is to derive the area and entropy spectra of the higher dimensional LDBHs by QNM method, which is prescribed by Maggiore [19]. As described in [29–31], here we use an approximation method in order to define the QNMs. Since the effective potential (8) vanishes at the horizon ($r^* \rightarrow -\infty$), therefore the QNMs are defined to be those for which one has purely ingoing plane wave at the horizon, namely,

$$\rho(r)|_{QNM} \sim e^{i\omega r^*} \text{ at } r^* \rightarrow -\infty, \quad (11)$$

Now we will solve equation (7) in the near horizon limit and then impose the above boundary condition to find the frequency of QNM i.e., ω .

Expansion of the metric function $f(r)$ around the event horizon yields,

$$\begin{aligned} f(r) &= f'(r_+(r-r_+)) + O[(r-r_+)^2] \\ &\simeq 2\kappa(r-r_+), \end{aligned} \quad (12)$$

where κ is the surface gravity, which is nothing but $\frac{1}{2}f'(r_+)$. Substituting this in the tortoise coordinate definition (9) and evaluating the integration one finds,

$$r^* \simeq \frac{1}{2\kappa} \ln(r-r_+), \quad (13)$$

Furthermore, substituting $\varepsilon = r - r_+$ in equations (8) and (12) and performing Taylor expansion around $\varepsilon = 0$ we obtain the near horizon form of the effective potential as,

$$V(\varepsilon) \simeq 2\kappa\varepsilon \left[\frac{l(l+N-3)}{A^2 r_+} \left(1 - \frac{\varepsilon}{r_+}\right) + \frac{(N-2)(N-6)}{8r_+^2} \kappa\varepsilon + \frac{(N-2)\kappa}{2r_+} \left(1 - \frac{\varepsilon}{r_+}\right) \right], \quad (14)$$

Also by substituting equation (13) into the Zerilli equation (7), we obtain the near horizon form of the Zerilli equation:

$$-4\kappa^2\varepsilon^2 \frac{d^2\rho(\varepsilon)}{d\varepsilon^2} - 4\kappa^2\varepsilon \frac{d\rho(\varepsilon)}{d\varepsilon} + V(\varepsilon)\rho(\varepsilon) = \omega^2\rho(\varepsilon), \quad (15)$$

Solution of the above equation yields,

$$\rho(\varepsilon) \sim \varepsilon^{\frac{i\omega}{2\kappa}} U(\hat{a}, \hat{b}, \hat{c}), \quad (16)$$

where $U(\hat{a}, \hat{b}, \hat{c})$ is the confluent hypergeometric function [32]. The parameters of the confluent hypergeometric functions are

$$\begin{aligned} \hat{a} &= \frac{1}{2} + i\left(\frac{\omega}{2\kappa} - \frac{\hat{\alpha}}{\hat{\beta}\sqrt{\kappa}A}\right), \\ \hat{b} &= 1 + i\frac{\omega}{\kappa}, \\ \hat{c} &= i\frac{\hat{\beta}\varepsilon}{2Ar_+\sqrt{\kappa}}, \end{aligned} \quad (17)$$

where

$$\begin{aligned} \hat{\beta} &= \sqrt{8l(l+N-3) - \kappa A^2(N-10)}, \\ \hat{\alpha} &= l(l+N-3) + \frac{N-2}{2}\kappa A^2, \end{aligned} \quad (18)$$

In the limit of $\varepsilon \ll 1$, the solution (16) reduces to the form

$$\rho(\varepsilon) \sim C_1 \varepsilon^{-\frac{i\omega}{2\kappa}} \frac{\Gamma(i\frac{\omega}{\kappa})}{\Gamma(\hat{a})} + C_2 \varepsilon^{\frac{i\omega}{2\kappa}} \frac{\Gamma(-i\frac{\omega}{\kappa})}{\Gamma(1 + \hat{a} - \hat{b})}, \quad (19)$$

where constants C_1 and C_2 denote the amplitudes of the near-horizon outgoing and ingoing waves, respectively. Now, since there is no outgoing wave in the QNM (11) at horizon, the first term should be vanished. This will happen at poles of the Gamma function of the denominator of the first term. The poles of the Gamma function will definitely determine the frequencies of the QNMs. Thus, one can read the frequencies of the QNMs of the higher dimensional LDBHs as,

$$\omega_n = \frac{2\sqrt{\kappa}\hat{\alpha}}{\hat{\beta}A} + i(2n+1)\kappa, \quad (20)$$

where $n = 1, 2, 3, \dots$. It is very important to note that the existence of the frequencies ω_n strictly depends on $\hat{\beta}$ parameter. It must certainly be real, which brings us a condition as follows

$$8l(l+N-3) > \kappa A^2(N-10), \quad (21)$$

Whenever the above condition holds, the imaginary part of the frequency of the QNM is

$$\omega_I = (2n+1)\kappa = \frac{2\pi}{\hbar}(2n+1)T_H, \quad (22)$$

where $T_H = \frac{\hbar\kappa}{2\pi}$ is so-called the Hawking temperature.

Now the energy of the such NAF black holes is calculated by the quasilocal mass definition [3]. Thus, one can determine the quasilocal mass of the higher dimensional LDBHs as,

$$M = \frac{r_+^{\frac{N-2}{2}}(N-2)\Sigma A^{N-2}}{8}, \quad (23)$$

Since from (22) $\Delta\omega = \omega_{n+1} - \omega_n = 4\pi T_H$, the adiabatic invariant quantity (2) in this case yields,

$$I_{adb} = \frac{\hbar}{4\pi} \int \frac{dM}{T_H}, \quad (24)$$

By considering the first law of thermodynamics, $T_H dS_{BH} = dM$, one can easily see that

$$I_{adb} = \frac{S_{BH}}{4\pi} \hbar, \quad (25)$$

Finally, according to the Bohr-Sommerfeld quantization rule

$$I_{adb} = \hbar n, \quad (26)$$

one gets the spacing of the entropy spectrum as

$$S_n = 4\pi n, \quad (27)$$

Recalling the relation $S = \frac{\mathcal{A}}{4}$, the area spectrum is given by

$$\mathcal{A}_n = 16\pi n, \quad (28)$$

with the spacing

$$\begin{aligned} \Delta S &= S_{n+1} - S_n = 4\pi \\ &= \frac{\Delta \mathcal{A}}{4}. \end{aligned} \quad (29)$$

We remark that both the entropy and area spectra are equally spaced and independent of the dimension of spacetime. This result is in agreement with that of Wei et al.'s conjecture [26]. As mentioned before, the above result is valid only for large quantum numbers n and the fulfilling of the condition (21).

IV. CONCLUSION

In this paper we investigated the area and entropy spectra of higher dimensional LDBHs. By utilizing the QNM method, it is showed the QNM frequencies become apparent with a particular condition $8l(l+N-3) > \kappa A^2(N-10)$. Unless this condition fails, the higher dimensional LDBHs have equally spaced area and entropy spectra. Both spectra are independent of the dimension of spacetime which means that our results in accordance with the conjecture of [26]. As a final remark, one should keep in mind that all calculations made here are semiclassical and based on Bohr-Sommerfeld quantization condition and QNMs. Finally, we want to point out that since the LDBHs are conformally related to the Brans-Dicke BHs [33], the same analysis might work for those BHs as well. This is going to be our next problem in the near future.

-
- [1] S.B. Giddings, A. Strominger, Phys. Rev. D **46**, 627 (1992).
 - [2] G. Clément, D. Gal'tsov, C. Leygnac, Phys. Rev. D **67**, 024012 (2003).
 - [3] J.D. Brown, J.W. York, Phys. Rev. D **47**, 1407 (1993).
 - [4] S.W. Hawking, Nature (London) **248**, 30 (1974).
 - [5] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
 - [6] K.C.K. Chan, J.H. Horne, R.B. Mann, Nucl. Phys. B **447**, 441 (1995).
 - [7] G.W. Gibbons and K. Maeda, Nucl. Phys. B **298**, 741 (1988).
 - [8] S.H. Mazharimousavi, M. Halilsoy, Phys. Lett. B **659**, 471 (2008).
 - [9] S.H. Mazharimousavi, M. Halilsoy, Z. Amirabi, Gen. Relativ. Gravit. **42**, 261 (2010).
 - [10] S.H. Mazharimousavi, I. Sakalli, M. Halilsoy, Phys. Lett. B **672**, 177 (2009).

- [11] J.D. Bekenstein, Lett. Nuovo Cimento **4**, 737 (1972).
- [12] J.D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
- [13] J.D. Bekenstein, Lett. Nuovo Cimento **11**, 467 (1974).
- [14] J.D. Bekenstein, gr-qc/9710076 (1997).
- [15] J.D. Bekenstein, gr-qc/9808028 (1998).
- [16] S. Hod, Phys. Rev. Lett. **81**, 4293 (1998).
- [17] R.L. Liboff, Found. Phys. Vol. **5**, No. 2, 271 (1975). doi: 10.1007/BF00717443
- [18] G. Kunstatter, Phys. Rev. Lett. **90**, 161301 (2003).
- [19] M. Maggiore, Phys. Rev. Lett. **100**, 141301 (2008).
- [20] E.C. Vagenas, J. High Energy Phys. **11**, 073 (2008).
- [21] A.J.M. Medved, Class. Quantum Gravity **25**, 205014 (2008). 0804.4346
- [22] W. Li, L. Xu, J. Lu, Phys. Lett. B **676**, 117 (2009).
- [23] D. Kothawala, T. Padmanabhan, S. Sarkar, Phys. Rev. D **78**, 104018 (2008).
- [24] A.L. Ortega, Phys. Lett. B **682**, 85 (2009).
- [25] M.R. Steare, D. Momeni, 1002.0185 (2010).
- [26] S.W. Wei, R. Li, Y.X. Liu, J.R. Ren, J. High Energy Phys. **03**, 076 (2009).
- [27] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, (Oxford University Press, USA, 1998).
- [28] D.P. Du, B. Wang, Phys. Rev. D **70**, 064024 (2004).
- [29] A.J. Medved, D. Martin, M. Visser, Class. Quantum Gravity **21**, 2393 (2004).
- [30] T. Padmanabhan, Class. Quantum. Gravity **21**, 2393 (2004).
- [31] T.R. Choudhury, T. Padmanabhan, Phys. Rev. D **69**, 064033 (2004).
- [32] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1965).
- [33] R.G. Cai, Y.S. Myung, Phys. Rev. D **56**, 3466 (1997). gr-qc/9702037.